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NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS AND SHARED VALUES*

BY

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ABSTRACT

Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a, b be two distinct finite complex numbers. If, for any $f \in \mathcal{F}$, f and f' share a IM, and $f'(z) = b$ whenever $f(z) = b$, then $\mathcal{F}$ is normal in D. This improves results due to Pang, Pang and Zalcman, Xu, etc.

1. Introduction

Let f be a nonconstant meromorphic function. In this paper, we use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), N(r,1/f), \ldots$$

(see Hayman [10], Schiff [18], Yang [23]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure.

Let g be a meromorphic function, and let a be a complex number. If $f - a$ and $g - a$ have the same zeros (ignoring multiplicity), then we say that f and g share value a IM.

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Let g be a meromorphic function in the complex plane. We say that g is a Yosida function if there exists a positive number M such that $g^{\#}(z) \leq M$ for all $z \in \mathbb{C}$, where

$$g^{\#}(z) = \frac{|g'(z)|}{1 + |g(z)|^2}$$

denotes the spherical derivative.

Gundersen [8] and Mues and Steinmetz [13] proved

THEOREM A: *Let a_1, a_2, a_3 be three distinct finite numbers, and let f be a nonconstant meromorphic function. If f and f' share a_1, a_2, a_3 IM, then $f(z) \equiv f'(z)$.*

Mues and Steinmetz [13] obtained

THEOREM B ([13]): *Let a, b be two distinct finite numbers, and let f be a nonconstant entire function. If f and f' share a, b IM, then $f(z) \equiv f'(z)$.*

Jank, Mues and Volkmann [12] proved

THEOREM C: *Let f be a nonconstant entire function, and let a be a nonzero finite complex number. If f and f' share a IM, and $f''(z) = a$ whenever $f(z) = a$, then $f(z) \equiv f'(z)$.*

Gundersen and Yang [9] and Al-khaladi [1] extended Theorem C as follows.

THEOREM D: *Let f be a nonconstant entire function with finite order, and let a be a nonzero finite complex number. If f and f' share a IM, and $f^{(k)}(z) = f^{(k+1)}(z) = a$ whenever $f(z) = a$, then $f(z) \equiv f'(z)$.*

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel (see Schiff [18]), if, for any sequence $f_n \in \mathcal{F}$, there exists a subsequence f_{n_j} such that f_{n_j} converges spherically locally uniformly to a meromorphic function or ∞ in D.

Schwick [19] first found the relation between normal families and shared values. He obtained the following normality criterion related to Theorem A.

THEOREM E: *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, a_3 be three distinct finite complex numbers. If, for any $f \in \mathcal{F}$, f and f' share a_1, a_2, a_3 IM, then $\mathcal{F}$ is normal in D.*

Xu [21, 22] and Pang [15] proved a normality criterion related to Theorem B.

THEOREM F: *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a, b be two distinct finite complex numbers. If, for any $f \in \mathcal{F}$, f and f' share a, b IM, then $\mathcal{F}$ is normal in D.*

In this paper, we improve Theorem F as follows.

THEOREM 1: *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a, b be two distinct finite complex numbers. If, for any $f \in \mathcal{F}$, f and f' share a IM, and $f'(z) = b$ whenever $f(z) = b$, then $\mathcal{F}$ is normal in D.*

THEOREM 2: *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a, b be two distinct finite complex numbers such that $b \neq 0$. If, for any $f \in \mathcal{F}$, f and f' share a IM, and $f(z) = b$ whenever $f'(z) = b$, then $\mathcal{F}$ is normal in D.*

Remark 1: Theorem 1 is not valid for a family of meromorphic functions.

Let $D = \{z\colon |z| < 1\}$, and

$$\mathcal{F} = \left\{ \frac{(2nz-1)^{2n}}{(2nz-1)^{2n}-1} \colon n = 1, 2, 3, \ldots \right\}.$$

Then for any $f \in \mathcal{F}$,

$$f = \frac{(2nz-1)^{2n}}{(2nz-1)^{2n}-1}, \quad f' = \frac{-4n^2(2nz-1)^{2n-1}}{[(2nz-1)^{2n}-1]^2}.$$

Obviously, the zeros of f are of multiplicity ≥ 2, f and f' have the same zeros, $f \neq 1$ (hence $f'(z) = 1$ whenever $f(z) = 1$).

On the other hand, we have

$$f^{\#}(0) = \frac{|f'(0)|}{1 + |f(0)|^2} = 4n^2 \to \infty, \quad \text{as } n \to \infty.$$

Hence by Marty's normality criterion, $\mathcal{F}$ is not normal in D.

Remark 2: $b \neq 0$ is necessary in Theorem 2.

Let $\mathcal{F} = \{e^{nz} - a/n + a\colon n = 1, 2, 3, \ldots\}$, $D = \{z\colon |z| < 1\}$. Then, for any $f \in \mathcal{F}$, $f(z) = e^{nz} - a/n + a$. Obviously, f and f' share a IM, $f'(z) \neq 0$. But $\mathcal{F}$ is not normal in D.

Remark 3: Pang and Zalcman [16] proved that Theorem F remains valid for a family of meromorphic functions. But Theorem 2 is not valid for a family of meromorphic functions.

Let a, b be two nonzero numbers such that $a = (m+1)b$, where m is a positive integer. Set $D = \{z\colon |z| < 1\}$, and

$$\mathcal{F} = \left\{ f_n(z) = b\left(z - \frac{1}{n}\right) + \frac{1}{m(nz-1)^m} + a\colon n = 1, 2, \ldots \right\}.$$

Then, for every $f \in \mathcal{F}$,

$$f(z) = b\left(z - \frac{1}{n}\right) + \frac{1}{m(nz-1)^m} + a, \quad f'(z) = b - \frac{n}{(nz-1)^{m+1}}.$$

Obviously, f and f' share a, $f'(z) \neq b$ (hence $f(z) = b$ whenever $f'(z) = b$). But $\mathcal{F}$ is not normal in D.

Chen and Hua [6] and Pang [15] obtained a normality criterion related to Theorem C.

THEOREM G: *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a be a nonzero finite complex number. If, for any $f \in \mathcal{F}$, f and f' share a IM, and $f''(z) = a$ whenever $f(z) = a$, then $\mathcal{F}$ is normal in D.*

Naturally, we ask whether there exists a normality criterion related to Theorem D ? In this paper, using the method of [17], which is different from [6, 21, 22], we give a positive answer to the question.

THEOREM 3: *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a be a nonzero finite complex number. If, for any $f \in \mathcal{F}$, f and f' share a IM, and $f^{(k)}(z) = a$, $f^{(k+1)}(z) = a$ whenever $f(z) = a$, then $\mathcal{F}$ is normal in D.*

2. Some lemmas

For convenience, we define

$$\begin{aligned} LD(r,f) :=& c_1 m\left(r, \frac{f'}{f}\right) + c_2 m\left(r, \frac{f'}{f-a}\right) + c_3 m\left(r, \frac{f''}{f'}\right) \\ &+ c_4 m\left(r, \frac{f''}{f'-a}\right) + c_5 m\left(r, \frac{f'''}{f''}\right), \quad a \in \mathbb{C} \end{aligned}$$

where $c_1, c_2, \ldots, c_5$ are constants, which may have different values at different occurrences. Let $LD(r, g)$ be similarly defined.

LEMMA 1: *Let f and g be nonconstant holomorphic functions on the unit disc Δ. Let*

$$P(z) = \frac{f''(z)}{f'(z)} - \frac{2f'(z)}{f(z)-1} - \frac{g''(z)}{g'(z)} + \frac{2g'(z)}{g(z)-1}.$$

If f and g share 1 IM on Δ, and $f(0) \neq 0, 1, f'(0) \neq 0, g(0) \neq 0, g'(0) \neq 0$ and $P(0) \neq 0$, then

$$T(r,f) \leq 4\overline{N}\left(r,\frac{1}{f}\right) + 3\overline{N}\left(r,\frac{1}{g}\right) + O(1)(LD(r,f)+LD(r,g)+1)$$
$$+ \log\frac{|f(0)(f(0)-1)|}{|f'(0)|} + 2\log\left|\frac{f(0)}{f'(0)}\right| + 2\log\left|\frac{g(0)}{g'(0)}\right| + \log\frac{1}{|P(0)|}.$$

Proof: By Nevanlinna's second fundamental theorem, we have

$$T(r,f)+T(r,g) \leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right)$$
$$- N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + O(1)(LD(r,f)+LD(r,g)+1)$$
$$+ \log\frac{|f(0)(f(0)-1)|}{|f'(0)|} + \log\frac{|g(0)(g(0)-1)|}{|g'(0)|}, \tag{2.1}$$

where $N_0(r, 1/f')$ denotes the counting function corresponding to the zeros of f' that are not zeros of $f(f-1)$; $N_0(r, 1/g')$ is defined similarly.

Let $N_L(r, 1/(f-1))$ and $\overline{N}_L(r, 1/(f-1))$ denote the counting function and the reduced counting function, respectively, for $1-$*points* of both f and g about which f has larger multiplicity than g, and $N_{11}(r, 1/(f-1))$ denote the counting function for common simple $1-$*points* of both f and g. Let $N_L(r, 1/(g-1))$ and $\overline{N}_L(r, 1/(g-1))$ be defined similarly.

Since f and g share 1 IM, we have

$$\overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right)$$
$$\leq N_{11}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + \overline{N}\left(r,\frac{1}{g-1}\right)$$
$$\leq N_{11}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right) + T(r,g) + \log\frac{1}{|g(0)-1|} + O(1).$$

Combining this and (2.1), we get

$$T(r,f) \leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + N_{11}\left(r,\frac{1}{f-1}\right) + \overline{N}_L\left(r,\frac{1}{f-1}\right)$$
$$- N_0\left(r,\frac{1}{f'}\right) - N_0\left(r,\frac{1}{g'}\right) + O(1)(LD(r,f)+LD(r,g)+1)$$
$$+ \log\frac{|f(0)(f(0)-1)|}{|f'(0)|} + \log\frac{|g(0)|}{|g'(0)|}. \tag{2.2}$$

Suppose that z_0 is a simple $1-$ *point* of both f and g. Then an elementary calculation shows that z_0 is a zero of P. Thus

$$\begin{aligned} N_{11}\left(r,\frac{1}{f-1}\right) \leq & N\left(r,\frac{1}{P}\right) \\ \leq & N(r,P)+O(1)(LD(r,f)+LD(r,g)+1)+\log\frac{1}{|P(0)|}. \end{aligned} \tag{2.3}$$

From the definition of $P(z)$, we have

$$\begin{aligned} N(r,P) \leq & \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right) \\ & +N_0\left(r,\frac{1}{f'}\right)+N_0\left(r,\frac{1}{g'}\right). \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} T(r,f) \leq & 2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+2\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right) \\ & +O(1)(LD(r,f)+LD(r,g)+1) \\ & +\log\frac{|f(0)(f(0)-1)|}{|f'(0)|}+\log\frac{|g(0)|}{|g'(0)|}+\log\frac{1}{|P(0)|}. \end{aligned} \tag{2.5}$$

Noting that

$$\begin{aligned} & N\left(r,\frac{1}{f-1}\right)-\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f}\right)-\overline{N}\left(r,\frac{1}{f}\right) \\ \leq & N\left(r,\frac{1}{f'}\right)\leq N\left(r,\frac{1}{f}\right)+LD(r,f)+\log\frac{|f(0)|}{|f'(0)|}, \end{aligned}$$

we have

$$\begin{aligned} N_L\left(r,\frac{1}{f-1}\right) \leq & N\left(r,\frac{1}{f-1}\right)-\overline{N}\left(r,\frac{1}{f-1}\right) \\ \leq & \overline{N}\left(r,\frac{1}{f}\right)+LD(r,f)+\log\frac{|f(0)|}{|f'(0)|}. \end{aligned} \tag{2.6}$$

Similarly, we have

$$N_L\left(r,\frac{1}{g-1}\right)\leq \overline{N}\left(r,\frac{1}{g}\right)+LD(r,g)+\log\frac{|g(0)|}{|g'(0)|}. \tag{2.7}$$

Substituting (2.6) and (2.7) in (2.5), we get the conclusion of Lemma 1. ∎

LEMMA 2: *Let f be a nonconstant holomorphic function on the unit disc Δ, and let a, b be two distinct nonzero complex numbers. If f and f' share a IM, $f'(z) = b$ whenever $f(z) = b$ on Δ, and $f(0) \neq a, b, f'(0) \neq 0, a, f''(0) \neq 0, f(0) \neq f'(0)$ and $Q[f(0)] \neq 0$, then*

$$\begin{aligned} &T(r,f)\\ \le &O(1)(LD(r,f)+1) + \log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)|} + 2\log\frac{|f(0)-b|}{|f''(0)|}\\ &+ 7\log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)(f(0)-f'(0))|} + 4\log\frac{|f'(0)(f'(0)-a)|}{|f''(0)(f(0)-f'(0))|} + \log\frac{1}{|Q[f(0)]|}, \end{aligned}$$

where

$$Q[f(z)] = \frac{f''(z)}{f'(z)} - \frac{2f'(z)}{f(z)-a} - \frac{f'''(z)}{f''(z)} + \frac{2f''(z)}{f'(z)-a}.$$

Proof: Since f and f' share a IM, $f'(z) = b$ whenever $f(z) = b$, we have

$$\begin{aligned} &N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-b}\right)\\ \le &N\left(r, \frac{1}{f-f'}\right) \le T(r, f-f') + \log\frac{1}{|f(0)-f'(0)|}\\ \le &T(r,f) + O(1)(LD(r,f)+1) + \log\frac{1}{|f(0)-f'(0)|}. \end{aligned} \tag{2.8}$$

On the other hand, it is easy to see that

$$m\left(r, \frac{1}{f-a}\right) + m\left(r, \frac{1}{f-b}\right) \le m\left(r, \frac{1}{f'}\right) + O(1)(LD(r,f)+1).$$

Combining this and (2.8), we get

$$\begin{aligned} &T\left(r, \frac{1}{f-a}\right) + T\left(r, \frac{1}{f-b}\right)\\ \le &T(r,f) + T\left(r, \frac{1}{f'}\right) - N\left(r, \frac{1}{f'}\right)\\ &+ O(1)(LD(r,f)+1) + \log\frac{1}{|f(0)-f'(0)|}. \end{aligned} \tag{2.9}$$

Thus we have

$$\begin{aligned} T(r,f) \le &T(r,f') - N\left(r, \frac{1}{f'}\right)\\ &+ O(1)(LD(r,f)+1) + \log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)(f(0)-f'(0))|}. \end{aligned} \tag{2.10}$$

Considering $T(r,f') \le T(r,f) + LD(r,f)$, we deduce from (2.10) that

$$N\left(r,\frac{1}{f'}\right) \le O(1)(LD(r,f)+1) + \log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)(f(0)-f'(0))|}. \tag{2.11}$$

Hence by (2.8), (2.10) and Nevanlinna's second theorem, we get

$$\begin{aligned}
N\left(r,\frac{1}{f-b}\right) =& N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) - N\left(r,\frac{1}{f-a}\right)\\
\le& T(r,f) - \overline{N}\left(r,\frac{1}{f-a}\right) + O(1)(LD(r,f)+1) + \log\frac{1}{|f(0)-f'(0)|}\\
\le& T(r,f') - N\left(r,\frac{1}{f'}\right) - \overline{N}\left(r,\frac{1}{f'-a}\right) + O(1)(LD(r,f)+1)\\
&+ \log\frac{1}{|f(0)-f'(0)|} + \log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)(f(0)-f'(0))|}\\
\le& O(1)(LD(r,f)+1) + \log\frac{|f'(0)(f'(0)-a)|}{|f''(0)|} + \log\frac{1}{|f(0)-f'(0)|}\\
&+ \log\frac{|(f(0)-a)(f(0)-b)|}{|f'(0)(f(0)-f'(0))|}.
\end{aligned} \tag{2.12}$$

Since f and f' share a IM, $(f-b)/(a-b)$ and f'/a share 1 IM. By applying Lemma 1 to the functions $(f-b)/(a-b)$ and f'/a, and using (2.11) and (2.12), we obtain the desired result. The proof of Lemma 2 is complete. ∎

LEMMA 3 (see Bureau [3]): *Let b_1, b_2 and b_3 be positive numbers and $U(r)$ a nonnegative, increasing and continuous function on $[r_0, R), R < \infty$. If*

$$U(r) \le b_1 + b_2 \log^+\frac{1}{\rho - r} + b_3 \log^+ U(\rho)$$

for any $r_0 < r < \rho < R$, then

$$U(r) \le B_1 + B_2 \log^+\frac{1}{R-r}$$

for $r_0 \le r < R$, where B_1 and B_2 depend on b_i $(i = 1,2,3)$ only.

LEMMA 4 (see Hiong [11]): *If $f(z)$ is meromorphic in $|z| < R$ and $f(0) \ne 0, \infty$, then for $0 < r < \rho < R$,*

$$\begin{aligned}
m\left(r,\frac{f^{(k)}}{f}\right) \le& O(1)\{1 + \log^+\log^+\frac{1}{|f(0)|} + \log^+\frac{1}{r}\\
&+ \log^+\frac{1}{\rho - r} + \log^+\rho + \log^+ T(\rho,f)\}.
\end{aligned}$$

The next lemma is a generalization of Zalcman's well-known lemma (see [24]).

LEMMA 5 (see [4, 17, 24, 25]): *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least k. If $\mathcal{F}$ is not normal at $z_0 = 0$, then for $0 \leq \alpha < k$, there exist*

(a) *a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$,*
(b) *a sequence of functions $f_n \in \mathcal{F}$, and*
(c) *a sequence of positive numbers $\rho_n \to 0$,*

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ converges locally uniformly to a nonconstant entire function g on $\mathbb{C}$. Moreover, g is of order at most one. If $\mathcal{F}$ possesses the additional property that there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for any $f \in \mathcal{F}$, then we can take $\alpha = k$.

LEMMA 6 ([10, 23]): *Let f be a nonconstant meromorphic function. Then for $k \geq 1$, $b \neq 0, \infty$,*

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - b}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).$$

3. Proof of Theorem 1

We distinguish three cases.

CASE 1: $ab \neq 0$. Without loss of generality, we may suppose that $D = \Delta = \{|z| < 1\}$, and prove normality at 0.

Let f_n be a sequence in $\mathcal{F}$, which is not normal at 0. We divide this case into two subcases.

CASE 1.1: $f'_n \equiv f_n$ for every $n \in \mathbb{N}$; then $f_n(z) \equiv C_n e^z$, which is normal at 0.

CASE 1.2: Consider the case that f_n and f'_n are not identical. By Lemma 5, after transition to a subsequence there are a sequence $z_n \to 0$, a positive sequence $\rho_n \to 0$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n \zeta)$$

tends to a nonconstant entire Yosida function g locally uniformly on compact subsets of $\mathbb{C}$. By Picard's Theorem, g must take on the value a or the value b (or both). If $g(\zeta_0) = a$, then Hurwitz's Theorem implies the existence of a sequence $\zeta_n \to \zeta_0$ with

$$f_n(z_n + \rho_n \zeta_n) = f'_n(z_n + \rho_n \zeta_n) = a.$$

This implies

$$g'(\zeta_0) = \lim_{n \to \infty} \rho_n f'_n(z_n + \rho_n \zeta_n) = 0.$$

Similarly, if $g(\zeta_1)=b$, then there is a sequence $\zeta_n'\to\zeta_1$ such that

$$f_n(z_n+\rho_n\zeta_n')=b,$$

hence

$$g'(\zeta_1)=\lim_{n\to\infty}\rho_n f_n'(z_n+\rho_n\zeta_n')=0.$$

Thus, in any case, g is not a polynomial of degree 1.

We claim that

$$3(g-a)g'(g'')^2-2(g')^3g''-(g')^2g'''(g-a)\not\equiv 0. \tag{3.0}$$

Indeed, otherwise $3(g-a)g'(g'')^2-2(g')^3g''-(g')^2g'''(g-a)\equiv 0$, so

$$3\frac{g''}{g'}-\frac{2g'}{g-a}-\frac{g'''}{g''}\equiv 0;$$

hence

$$(g')^3=C(g-a)^2g'', \tag{3.1}$$

where C is a non-zero constant.

Suppose now that $g(\zeta_0)=a$; then $g'(\zeta_0)=0$. Now (3.1) implies $g\neq a$. By a result of Clunie and Hayman [7], the order of the entire Yosida function g is one. So

$$g(\zeta)=a+e^{c\zeta+d}, \tag{3.2}$$

where c ($\neq 0$) and d are complex numbers. Since $g\neq a$, there exists ζ_1 such that $g(\zeta_1)=b$, $g'(\zeta_1)=0$. But this contradicts (3.2). It follows that (3.1) fails, so that (3.0) holds.

Choose a point $\zeta_0\in\mathbb{C}$ such that

$$g(\zeta_0)\neq a,b;\quad g'(\zeta_0)\neq 0,\ g''(\zeta_0)\neq 0, \tag{3.3}$$

and

$$\begin{aligned}&3(g(\zeta_0)-a)g'(\zeta_0)(g''(\zeta_0))^2-2(g'(\zeta_0))^3g''(\zeta_0)\\ &-(g'(\zeta_0))^2g'''(\zeta_0)(g(\zeta_0)-a)\neq 0;\end{aligned} \tag{3.4}$$

then

$$\frac{1}{\rho_n}\frac{(f_n(z_n+\rho_n\zeta_0)-a)(f_n(z_n+\rho_n\zeta_0)-b)}{f_n'(z_n+\rho_n\zeta_0)}\to\frac{(g(\zeta_0)-a)(g(\zeta_0)-b)}{g'(\zeta_0)},$$

$$\frac{1}{\rho_n^2}\frac{f_n(z_n+\rho_n\zeta_0)-b}{f_n''(z_n+\rho_n\zeta_0)}\to\frac{g(\zeta_0)-b}{g''(\zeta_0)},$$

$$\frac{1}{\rho_n^2}\frac{(f_n(z_n+\rho_n\zeta_0)-a)(f_n(z_n+\rho_n\zeta_0)-b)}{f_n'(z_n+\rho_n\zeta_0)(f_n(z_n+\rho_n\zeta_0)-f_n'(z_n+\rho_n\zeta_0))}\to-\frac{(g(\zeta_0)-a)(g(\zeta_0))-b)}{(g'(\zeta_0))^2},$$

and

$$\frac{1}{\rho_n}\frac{f_n'(z_n+\rho_n\zeta_0)(f_n'(z_n+\rho_n\zeta_0)-a)}{f_n''(z_n+\rho_n\zeta_0)(f_n(z_n+\rho_n\zeta_0)-f_n'(z_n+\rho_n\zeta_0))}\to -\frac{(g'(\zeta_0))^2}{g'(\zeta_0)g''(\zeta_0)}.$$

Set $Q_n = Q(f_n)$. Since

$$\frac{1}{Q}=\frac{(f-a)(f'-a)f'f''}{P(f,f',f'',f''')},$$

where

$$\begin{aligned}P(f,f',f'',f''') =&3(f-a)f'(f'')^2-2(f')^3f''-(f')^2f'''(f-a)\\&+[2a(f')^2f''+a(f-a)f'f'''-a(f-a)(f'')^2],\end{aligned}$$

then

$$\frac{1}{\rho_n}\frac{1}{Q_n(z_n+\rho_n\zeta_0)}\to\frac{(g(\zeta_0)-a)(g'(\zeta_0))^2g''(\zeta_0)}{R(g(\zeta_0),g'(\zeta_0),g''(\zeta_0),g'''(\zeta_0))}, \tag{3.5}$$

where

$$\begin{aligned}&R(g(\zeta_0),g'(\zeta_0),g''(\zeta_0),g'''(\zeta_0))\\&=3(g(\zeta_0)-a)g'(\zeta_0)(g''(\zeta_0))^2-2(g'(\zeta_0))^3g''(\zeta_0)-(g'(\zeta_0))^2g'''(\zeta_0)(g(\zeta_0)-a).\end{aligned}$$

Therefore we have

$$\log\frac{|(f_n(z_n+\rho_n\zeta_0)-a)(f_n(z_n+\rho_n\zeta_0)-b)|}{|f_n'(z_n+\rho_n\zeta_0)|}\to-\infty, \tag{3.6}$$

$$\log\frac{|f_n(z_n+\rho_n\zeta_0)-b|}{|f_n''(z_n+\rho_n\zeta_0)|}\to-\infty, \tag{3.7}$$

$$\log\frac{|(f_n(z_n+\rho_n\zeta_0)-a)(f_n(z_n+\rho_n\zeta_0)-b)|}{|f_n'(z_n+\rho_n\zeta_0)(f_n(z_n+\rho_n\zeta_0)-f_n'(z_n+\rho_n\zeta_0))|}\to-\infty, \tag{3.8}$$

$$\log\frac{|f_n'(z_n+\rho_n\zeta_0)(f_n'(z_n+\rho_n\zeta_0)-a)|}{|f_n''(z_n+\rho_n\zeta_0)(f_n(z_n+\rho_n\zeta_0)-f_n'(z_n+\rho_n\zeta_0))|}\to-\infty, \tag{3.9}$$

and

$$\log\frac{1}{|Q_n(z_n+\rho_n\zeta_0)|}\to-\infty. \tag{3.10}$$

For $n=1,2,\ldots$, put

$$p_n(z)=f_n(z_n+\rho_n\zeta_0+z).$$

Let n be sufficiently large; then $p_n(z)$ is defined and holomorphic on the disk $0 < |z| < \frac{1}{2}$, since $z_n + \rho_n\zeta_0 \to 0$. By (3.3), (3.4) and (3.5) we have

$$\begin{aligned} p_n(0) &= g_n(\zeta_0) \to g(\zeta_0) \neq a, b; \\ p_n'(0) &= \frac{1}{\rho_n} g_n'(\zeta_0) \to \infty; \\ p_n''(0) &= \frac{1}{\rho_n^2} g_n''(\zeta_0) \to \infty; \\ Q(p_n(0)) &= Q_n(z_n + \rho_n\zeta_0) \to \infty. \end{aligned}$$

Therefore, we may apply Lemma 2 to $p_n(z)$ and, using (3.6)–(3.10), we obtain

$$T(r, p_n) \leq O(1)LD(r, p_n),$$

for sufficiently large n. Hence by Lemma 3 and Lemma 4, we can get

$$T\left(\frac{1}{4}, p_n\right) \leq C,$$

where C is a constant independent of n. Thus $f_n(z)$ is bounded for sufficiently large n and $|z| < \frac{1}{8}$. We arrive at a contradiction. The proof of Case 1 is complete.

CASE 2: $a = 0, b \neq 0$.

Without loss of generality, we may assume that $b = 1$. Then, using almost the same argument as in [21, 22], we can prove this case. Here we omit the details.

CASE 3: $a \neq 0, b = 0$.

Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z_0 \in D$. Since $f'(z) = 0$ whenever $f(z) = 0$, f has only multiple zeros. Applying Lemma 5 to $\alpha = 1$, we obtain a sequence of functions $f_n \in \mathcal{F}$, a sequence $z_n \to z_0$ and a positive sequence $\rho_n \to 0$, such that

$$g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n\zeta) \to g(\zeta)$$

locally uniformly, where $g(\zeta)$ is a nonconstant entire function. Moreover, g is of order at most one.

Since f has only zeros of multiplicity at least 2, then by Hurwitz's Theorem, g has no zeros of multiplicity less than 2. We claim that there exists ζ_0 such that $g'(\zeta_0) = a$.

Indeed, suppose that $g'(\zeta) \neq a$; then we have

$$g'(\zeta) = a + c_1 e^{c_2\zeta}. \tag{3.11}$$

Thus we get

$$g(\zeta) = a\zeta + c_3 + \frac{c_1}{c_2} e^{c_2\zeta}. \tag{3.12}$$

It follows that almost all the zeros of g are simple, which contradicts the fact that g has no zeros of multiplicity less than 2. Obviously, $g(\zeta) \not\equiv a$. Then there exist ζ_n, $\zeta_n \to \zeta_0$, such that

$$f'_n(z_n + \rho_n\zeta_n) = g'_n(\zeta_n) = a,$$

for sufficiently large n. Since f_n and f'_n share a IM, we have $f_n(z_n + \rho_n\zeta_n) = a$. Thus by $g_n(\zeta_n) = \rho_n^{-1} f_n(z_n + \rho_n\zeta_n) = \rho_n^{-1} a$ we obtain that $g(\zeta_0) = \infty$, a contradiction. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

We assume that $D = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D; without loss of generality we assume that $\mathcal{F}$ is not normal at the point $z_0 = 0$. Then by Lemma 5, there exist

(a) a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$,
(b) a sequence of functions $f_n \in \mathcal{F}$, and
(c) a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\xi) = \rho_n^{-1}[f_n(z_n + \rho_n\xi) - a]$ converges locally uniformly to a non-constant entire function g. Moreover, g is of order at most one.

CLAIM: *$g = 0$ if and only if $g' = a$, and $g'(\xi) \neq b$.*

Indeed, suppose that $g(\xi_0) = 0$. Then by Hurwitz's Theorem, there exist ξ_n, $\xi_n \to \xi_0$ such that

$$g_n(\xi_n) = \rho_n^{-1}[f_n(z_n + \rho_n\xi_n) - a] = 0.$$

Thus $f_n(z_n + \rho_n\xi_n) = a$. Since f_n and f'_n share a IM, we have

$$g'_n(\xi_n) = f'_n(z_n + \rho_n\xi_n) = a.$$

Hence $g'(\xi_0) = \lim_{n\to\infty} g'_n(\xi_n) = a$. Thus we have proved that $g' = a$ whenever $g = 0$.

On the other hand, if $g'(\xi_0) = a$, then there exist ξ_n, $\xi_n \to \xi_0$, such that

$$g'_n(\xi_n) = f'_n(z_n + \rho_n\xi_n) = a, \quad n = 1, 2, \ldots.$$

Hence $f_n(z_n + \rho_n\xi_n) = a$ and $g_n(\xi_n) = 0$ for $n = 1, 2, \ldots$, since f_n and f_n' share a IM. Thus $g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n) = 0$. This shows that $g = 0$ whenever $g' = a$. Hence $g = 0$ if and only if $g' = a$.

Next we prove that $g'(\xi) \neq b$. Suppose that there exists ξ_0 satisfying $g'(\xi_0) = b$. Then, by Hurwitz's Theorem, there exists a sequence ξ_n such that $\xi_n \to \xi_0$ and $g_n'(\xi_n) = f_n'(z_n + \rho_n\xi_n) = b$ for $n = 1, 2, \ldots$. Since $f_n(z) = b$ whenever $f_n'(z) = b$, we have $f_n(z_n+\rho_n\xi_n) = b$ and $g_n(\xi_n) = \rho_n^{-1}[f_n(z_n+\rho_n\xi_n)-a] = \rho_n^{-1}(b-a) \to \infty$. This contradicts $\lim_{n\to\infty} g_n(\xi_n) = g(\xi_0) \neq \infty$. So, $g'(\xi) \neq b$.

Hence we get

$$g'(\xi) = b + e^{A\xi+B}. \tag{4.1}$$

We claim that $A = 0$. Suppose that $A \neq 0$; then

$$g(\xi) = b\xi + \frac{e^{A\xi+B}}{A} + C. \tag{4.2}$$

Let $g'(\xi) = a$. Then by (4.1), (4.2) and $g'(\xi) = a$ implies $g(\xi) = 0$, we deduce that $\xi = -(CA + a - b)/(bA)$, which contradicts that $g'(\xi) = a$ has infinitely many solutions. Thus we have

$$g'(\xi) = b + e^B, \quad g(\xi) = (b + e^B)\xi + C. \tag{4.3}$$

Since g is nonconstant, this contradicts the fact that $g(\xi) = 0$ if and only if $g'(\xi) = a$. Thus $\mathcal{F}$ is normal in D. The proof of Theorem 2 is complete.

5. Proof of Theorem 3

We assume that $D = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D; without loss of generality we assume that $\mathcal{F}$ is not normal at the point $z_0 = 0$. Then by Lemma 5, there exist

(a) a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$,

(b) a sequence of functions $f_n \in \mathcal{F}$, and

(c) a sequence of positive numbers $\rho_n \to 0$,

such that $g_n(\xi) = \rho_n^{-1}[f_n(z_n + \rho_n\xi) - a]$ converges locally uniformly to a non-constant entire function g.

Next, using the same reasoning as in the proof of Theorem 2, we can prove that $g = 0$ if and only if $g' = a$, and $g^{(k)} = g^{(k+1)} = 0$ whenever $g = 0$ for $k \geq 2$.

Now we consider two cases.

CASE 1: $k \geq 2$. Then by Lemma 6 and Nevanlinna's first fundamental theorem, we have

$$\begin{aligned} T(r,g) \leq& N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g'-a}\right)-N\left(r,\frac{1}{g^{(k)}}\right)+S(r,g)\\ \leq& N\left(r,\frac{1}{g'-a}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,g)\\ \leq& T\left(r,\frac{1}{g'-a}\right)-\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,g)\\ \leq& T(r,g'-a)-\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,g)\\ \leq& T(r,g)-\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,g). \end{aligned} \tag{5.1}$$

Thus we get

$$\overline{N}\left(r,\frac{1}{g^{(k)}}\right)=S(r,g). \tag{5.2}$$

Hence by (5.1), (5.2) and the claim ($g=0$ if and only if $g'=a$, $g^{(k)}=g^{(k+1)}=0$ whenever $g=0$) we get a contradiction: $T(r,g)=S(r,g)$.

CASE 2: $k=1$. Let $\xi_0 \in \mathbb{C}$ such that $g(\xi_0)=0$. Then $g'(\xi_0)=a$, $g''(\xi_0)=0$. Thus ξ_0 is a multiple zero of $g'(\xi)-a$, so by Hurwitz's Theorem there exist two sequences $\{\xi_{in}\}, i=1,2$, such that $\lim_{n\to\infty}\xi_{in}=\xi_0$, and for large n,

$$g_n'(\xi_{1n})=g_n'(\xi_{2n})=a.$$

Hence, since f_n and f_n' share a IM, we get

$$g_n(\xi_{1n})=g_n(\xi_{2n})=0. \tag{5.3}$$

Thus by $f''=a$ whenever $f=a$, we have

$$g_n''(\xi_{in})=\rho_n f_n''(z_n+\rho_n\xi_{in})\neq 0, \quad i=1,2,$$

so each ξ_{in} is a simple zero of $g_n'-a$. Hence we deduce that $\xi_{1n}\neq\xi_{2n}$ for $n=1,2,3,\ldots$. Thus by Hurwitz's Theorem, ξ_0 is a multiple zero of $g(\xi)$, which contradicts $g'(\xi_0)=a\neq 0$.

Hence $\mathcal{F}$ is normal in D. This completes the proof of Theorem 3.

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